

An explicit connection between the basis states $|n\rangle \equiv |n_1, n_2, \dots\rangle$ with $\sum_\nu n_\nu = N$ in second quantization and the basis wavefunctions $\Phi_n(x_1, \dots, x_N)$ in first quantization can be established. Let

$$|x_1, x_2, \dots, x_N\rangle \equiv \frac{1}{\sqrt{N!}} \hat{\psi}^\dagger(x_1) \dots \hat{\psi}^\dagger(x_N) |0\rangle. \quad (1)$$

Then the (correctly normalized) basis wavefunctions are given by

$$\Phi_n(x_1, \dots, x_N) = \langle x_1, \dots, x_N | n \rangle. \quad (2)$$

As an example, consider a fermionic 2-particle basis state $|\dots, 1_\mu, \dots, 1_\nu, \dots\rangle$ in which the single-particle states μ and ν are occupied and all others are empty. Evaluate the rhs of (2) to show that the wavefunction is indeed the correct Slater determinant,

$$\Phi(x_1, x_2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} \phi_\mu(x_1) & \phi_\mu(x_2) \\ \phi_\nu(x_1) & \phi_\nu(x_2) \end{vmatrix}. \quad (3)$$

Consider a bosonic particle moving in 1D with the Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) + \hbar\omega_0 \left(a^\dagger + a \right),$$

where $[a, a^\dagger] = 1$, while ω and ω_0 are positive constants. Diagonalize H by introducing the operator $\alpha \equiv a + \omega_0/\omega$ and its Hermitian conjugate α^\dagger , and determine the eigenenergies. What might be the physical origin of the second term in H (see Section 1.4.1)? Compare the result to a classical and a first quantized treatment of the problem.

Starting from Eqs. (2.34) and (2.45) derive the expression Eq. (2.47) for the contributions from the direct Coulomb interaction processes to the interaction energy in second order perturbation theory.

In Section 2.3.2 we saw an example of the existence of 2D electron gases in GaAs–Ga_{1–x}Al_xAs heterostructures. Derive, in analogy with the 3D case, the relation between the 2D Fermi wave vector k_F and the 2D electron density: $k_F^2 = 2\pi n$. Use the result to derive the 2D density of states per area, $d(\varepsilon)$.

Calculate in the harmonic approximation the internal energy $U(T) = \langle H_{\text{ph}} \rangle$ ($\langle \dots \rangle$: thermal average) of the lattice vibrations of a three-dimensional crystal. Discuss the limiting cases of high and low temperatures (Hint: $\langle b_{\mathbf{k}\lambda}^\dagger b_{\mathbf{k}\lambda} \rangle \Rightarrow$ Bose-Einstein distribution).

$$H_{\text{ph}} = \sum_{\mathbf{k}\lambda} \hbar\omega_{\mathbf{k}\lambda} \left(b_{\mathbf{k}\lambda}^\dagger b_{\mathbf{k}\lambda} + \frac{1}{2} \right), \quad [b_{\mathbf{k}\lambda}, b_{\mathbf{k}'\lambda'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'}. \quad (3.23)$$

In the jellium model of metals the kinetic energy of the electrons is described by the Hamiltonian H_{jel} of Eq. (2.19), while the interaction energy is given by $V'_{\text{el-el}}$ of Eq. (2.34). In the Heisenberg picture the time evolution of the electron creation and annihilation operators $c_{\mathbf{k}\sigma}^\dagger$ and $c_{\mathbf{k}\sigma}$ is governed by the total Hamiltonian $H = H_{\text{jel}} + V'_{\text{el-el}}$. In analogy with Eq. (5.31) derive the equation of motion for $c_{\mathbf{k}\sigma}(t)$. Apply the Hartree–Fock approximation to the result.